Mathematical Modeling of Radial Flow Filtration and Its Application to Groundwater Recharge Problems

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INTRODUCTION

The problem of radial flow filtration has been recognized in groundwater recharge through boreholes and water injection during secondary recovery of petrol. During outward radial flow, the suspension experiences decreasing velocities as it passes through the filter media, whereas in vertical downward filtration, suspension flows at constant velocity. Therefore in this study, O'Melia's model for deep-bed filtration (O'Melia and Ali, 1978) has been modified to radial flow conditions by considering the velocity variation along the radius. This model was then verified with bench-scale experimental results. An example simulation study to calculate the concentration and headloss profile in an aquifer during recharge has been illustrated.

MATHEMATICAL MODELING OF RADIAL FILTER

The single collector efficiency (η_r), from its definition, can be expressed as follows:

$$\eta_r = \frac{\text{Rate at which particles strike and attach to a filter grain} + N \cdot \text{rate at which particles strike and attach to a retained particle which act as particle collector}}{\text{Rate at which particles approach the collector}}$$

$$= \frac{\alpha\eta\left[\frac{\pi}{4}d_c^2 \cdot v(r) \cdot n(r,t)\right] + N(r,t)\cdot\eta_p\alpha_p\left[\frac{\pi}{4}d_p^2 \cdot v(r)\cdot n(r,t)\right]}{\frac{\pi}{4}d_c^2\, v(r)n(r,t)}$$

$$= \alpha\eta + \alpha_p\eta_p\left(\frac{d_p}{d_c}\right)^2 N(r,t) \tag{1}$$

The velocity variation along the radius is given by

$$v(r) = \frac{Q}{2\pi r \cdot H} \tag{2}$$

If one considers the rate of change of particle collectors equal to a fraction of total number of retained particles, then the following relationship can be established:

$$\frac{\partial N(r,t)}{\partial t} = \beta \cdot \eta\alpha \cdot v(r)n(r,t)\cdot\frac{\pi d_c^2}{4} \tag{3}$$

Taking the mass balance of suspended particles for a ring of filter media of thickness ΔL and radius r:

Accumulation = Input − Output − Removal

$$\therefore (2\pi r dr \cdot \Delta L)\frac{\partial n(r,t)}{\partial t} = 2\pi r \Delta L \cdot v(r)n(r,t) - (2\pi r\cdot\Delta L + 2\pi dr\cdot\Delta L)\left(v(r) + \frac{\partial v(r)}{\partial r}dr\right)\left(n(r,t) + \frac{\partial n(r,t)}{\partial r}dr\right) - \frac{3}{2}(2\pi r\cdot dr\cdot\Delta L)\cdot\frac{(1-f)}{d_c}\cdot v(r)\cdot n(r,t)\eta_r(r,t) \tag{4}$$

Neglecting second-order terms and simplifying, one can write Eq. 4 as

$$\frac{\partial n(r,t)}{\partial t} + \frac{\partial}{\partial r}[v(r)\cdot n(r,t)] + \frac{1}{r}\cdot n(r,t)\cdot v(r) + \frac{3}{2}\cdot\frac{(1-f)}{d_c}\cdot\eta_r(r,t)\cdot v(r)\cdot n(r,t) = 0 \tag{5}$$

Combining Eqs. 1, 2, 3, and 5

$$\frac{\partial^2\eta_r(r,t)}{\partial t^2} + \frac{C_1}{r}\frac{\partial^2\eta_r(r,t)}{\partial r\cdot\partial t} + \frac{C_1}{r^2}\frac{\partial n_r(r,t)}{\partial t} + \frac{DC_1}{r}\cdot\eta_r(r,t)\cdot\frac{\partial\eta_r}{\partial t}(r,t) = 0 \quad (6)$$

in which

$$C_1 = Q/2\pi H$$

and

$$D = \frac{3}{2}\frac{(1-f)}{d_c}$$

Calculation of η_r

The removal efficiency of the collector η_r at different times and filter radius can be calculated from Eq. 6 using the following boundary conditions:

$$\eta_r(o,t) = \alpha\eta + \alpha_p\beta\cdot\eta_p\cdot\alpha\eta\cdot v\cdot\frac{\pi}{4}\cdot d_p^2\cdot n_o\cdot(t-\Delta t) \quad (7)$$

$$\eta_r(r,o) = \frac{\alpha\eta}{v(r)} \quad (8)$$

Since there is no analytical solution for Eq. 6, it has been solved numerically using forward finite-difference method.

Calculation of Clean Bed Removal Efficiency

From the mass balance for suspended solids removal

$$\frac{Adr(1-f)}{\frac{\pi}{6}d_c^3}\cdot\left(\frac{\pi}{4}d_c^2\right)\cdot v(r)\cdot c(r,t)\cdot\eta_r(r,t) = -Av(r)\cdot dc(r,t) \quad (9)$$

i.e.

$$-\frac{3}{2}\frac{(1-f)}{d_c}\eta_r(r,t)dr = -\frac{dc(r,t)}{c(r,t)}$$

at $t = 0$

$$\frac{3}{2}\frac{(1-f)}{d_c}\eta_r(r,o)dr = \frac{-dc(r,o)}{c(r,o)} \quad (10)$$

Here, $\eta_r(r,o)$ is a function of velocity (Ives, 1971) and is inversely proportional to the velocity; i.e.

$$\eta_r(r,o) = \frac{\alpha\eta}{v(r)}$$

This also can be written:

$$\eta_r(\bar{r},o) = \left(\frac{\alpha\eta}{v_o}\right)\bar{r} \quad (11)$$

in which

$$\bar{r} = \frac{r}{r_o}$$

and

$$v_o = \text{approach velocity} = \frac{Q}{2\pi r_o H}$$

Substituting Eq. 11 into Eq. 10, one obtains

$$\frac{3}{2}\frac{(1-f)}{d_c}\frac{\alpha\eta}{v_o}\cdot\bar{r}\cdot r_o\cdot dr = \frac{-dc(r,t)}{c(r,t)}$$

Integrating the above equation and rearranging yields

$$\eta_r(r_o,o) = \frac{\alpha\eta}{v_o} = -\ln\left(\frac{c^*}{c_o}\right)\cdot\frac{2}{3}\cdot\frac{d_c}{(1-f)}\cdot\frac{2}{r_o(\bar{r}^2-1)}$$

in which c^* = effluent concentration at $t = 0$.

The above equation can also be written in the following form:

$$\eta_r(\bar{r},o) = -\frac{2}{3}\ln\left(\frac{c^*}{c_o}\right)\cdot\frac{d_c}{(1-f)}\cdot\frac{2\cdot\bar{r}}{r_o(\bar{r}^2-1)} \quad (12)$$

Calculation of Local Concentration

From the mass balance for suspended solids removal (Eq. 9), the following equation can be derived:

$$c(r+dr,t) = c(r,t) - c(r,t)\cdot\frac{3}{2}\frac{(1-f)}{d_c}\eta_r(r,t)\cdot dr \quad (13)$$

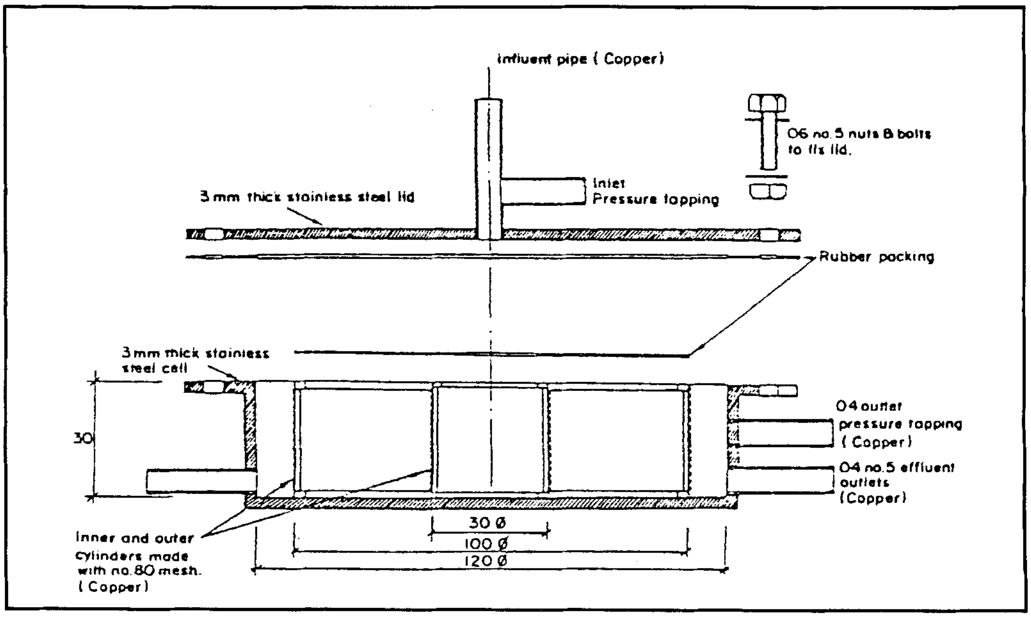


Figure 1. Radial filter, laboratory model (dimensions in mm).

TABLE 1. VALUES OF $\alpha_p\beta$ AND β' FOR DIFFERENT FILTER OPERATING CONDITIONS

c_o, mg/L	V_0, m^3/m^2·h	d_c, mm	$\alpha_p\beta$	β'
		Influence of Influent Concentration		
25	10	0.595–0.841	0.490	1
50			0.210	1
100			0.096	0.80
150			0.064	0.68
		Influence of Filtration Rate		
100	5	0.595–0.841	0.0950	1
	10		0.0960	0.80
	15		0.0950	0.72
	20		0.120	0.64
		Influence of Filter Grain Size		
100	10	0.595–0.841	0.096	0.80
		0.841–1.00	0.096	0.52
		1.00–1.19	0.064	0.52
		1.19–1.39	0.040	0.52

Therefore, using the above equation with the following boundary condition, one could compute the concentration at any time t and filter thickness r:

$$c(r_o,t) = c_o \tag{14}$$

Headloss Computation

From Darcy's equation

$$v(r) = B \cdot \frac{dh}{dr} \tag{15}$$

in which B is the permeability and is given by the following expression:

$$1/B = \frac{k\mu}{\rho_l \cdot g}\frac{(1-f)^2}{f^3}a_g^2 \tag{16}$$

Here K = Kozney's constant and a_g = specific surface.

The specific surface can be calculated from the following formula:

$$a_g^2 = \frac{36}{d_c^2}\left[\frac{1+\beta'\left(\frac{N_p}{N_c}\right)\left(\frac{d_p}{d_c}\right)^2\left(\frac{S}{6.0}\right)}{1+\left(\frac{N_p}{N_c}\right)\left(\frac{d_p}{d_c}\right)^3}\right]^2 \tag{17}$$

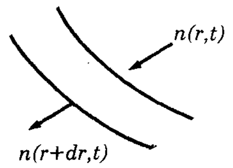
Substituting Eqs. 16 and 17 into 15, one obtains

$$dh = 36K\frac{\mu}{\rho_l g}\frac{(1-f)^2}{f^3}\frac{1}{d_c^2} \cdot v(r)\left[\frac{1+\beta'\left(\frac{N_p}{N_c}\right)\left(\frac{d_p}{d_c}\right)^2\left(\frac{S}{6.0}\right)}{1+\left(\frac{N_p}{N_c}\right)\left(\frac{d_p}{d_c}\right)^3}\right]dr \tag{18}$$

Headloss increments found in each layer from Eq. 18 are summed up to calculate the cumulative headloss. The main variable parameter that contributes to the headloss development is $\beta' N_p$, a fraction of the total number of particles deposited in the bed. The total number of deposited particles can be calculated as described in the following section, whereas β' is calculated by fitting the theoretical headloss computed at different β' values, with experimental headloss profiles.

Calculation of Total Number of Deposited Particles

Considering a layer of filter media of thickness dr,



Total number of particles deposited up to time $t + dt$
= Total number of particles deposited up to time t + number of particles deposited between time t and $t + dt$.

Assuming the concentration during the time increment of dt to remain constant, the total number of deposited particles can be calculated in the following way:

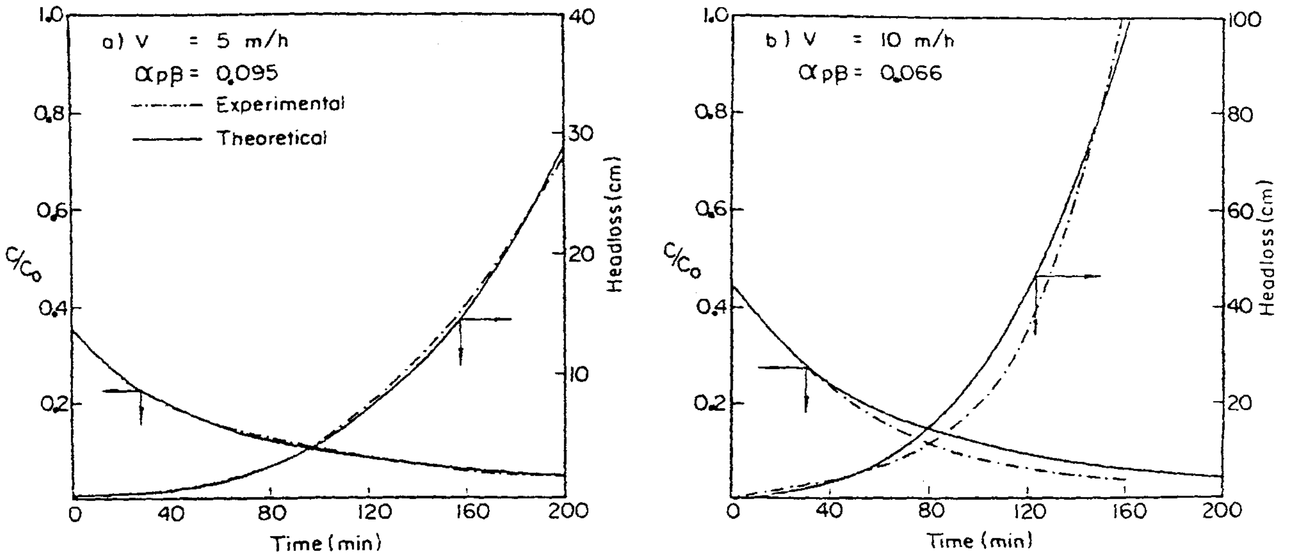


Figure 2. Influence of filtration rate.

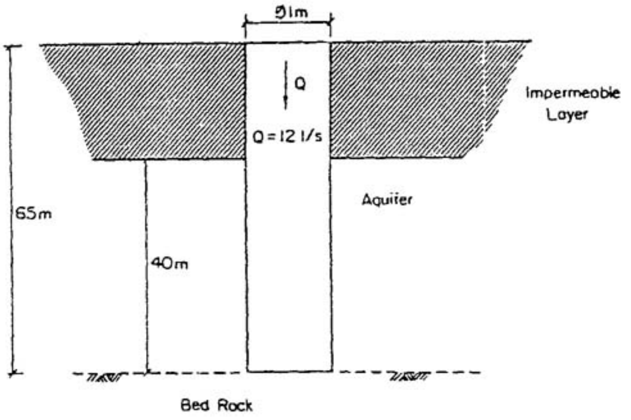


Figure 3. Recharge problem considered.

$$N_p(r,t + dt) = N_p(r,t) + [n(r,t) - n(r + dr,t)] \cdot v(r) \cdot A \cdot dt \qquad (19)$$

The variations of $v(r)$ and A in each layer are taken into account.

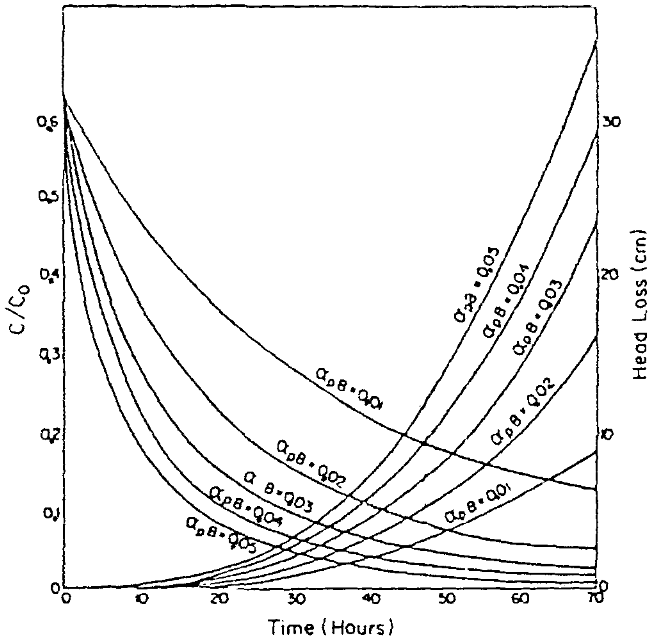


Figure 4. Simulated concentration and headloss profiles.

EXPERIMENTAL

The laboratory-scale radial filter model used in this study is as shown in Figure 1. An artificial suspension of kaolin clay ($d_{60} \simeq 4\mu$) was used at known concentrations. Narrowly graded sand of known size ranges was used as the filter medium. A cationic polymer, Zetag 32 (Allied Colloids, Ltd., U.K.), at a dose of 0.05 mg/L (optimum dose) was added continuously to the filter, in order to increase the attachment of particles with filter medium. The filter performance was measured throughout the filter run in terms of suspended solids removal efficiency and headloss development.

Verification of the Mathematical Model

The mathematical model was verified using the experimental results of the laboratory-scale radial flow filter. The values of $\alpha_p\beta$ used to fit the theoretical curves with the experimental concentration profiles are presented in Table 1. For better illustration, the curves obtained at two different filtration rates are presented in Figure 2. Here α_p and β are two unknown parameters in the mathematical model and they are found in a product form. This $\alpha_p\beta$ value was calculated in a way similar to that of O'Melia in his deep-bed filter analysis (O'Melia and Ali, 1978). The parameter β' appearing in the headloss equation was calculated by fitting the theoretical headloss profiles computed at different β' values with experimental headloss profiles. The β' values calculated for different filtration conditions are summarized in Table 1.

It can be seen from Table 1 that the concentration and headloss profiles obtained for different filtration conditions could not be fitted with same values of $\alpha_p\beta$ and β'. This is the major shortcoming of this model. The same phenomenon was also observed by Tobiason (1979) and Vigneswaran (1980) in their deep-bed filter analysis.

Practical Application of this Mathematical Model

This mathematical formulation can be used to predict the particle removal and the extent of clogging during artificial recharge in an aquifer under ideal flow conditions (with the help of limited number of laboratory-scale radial flow filter experiments performed with the same sand and recharging water). As an example, the following recharge problem was considered, the dimensions of which are given in Figure 3. The simulated c/c_o and headloss profiles for this recharge problem at different $\alpha_p\beta$ values are summarized in Figure 4.

CONCLUSION

The modified O'Melia model was able to predict the performance of the radial filter. However, the concentration and headloss profiles obtained at different filtration rates, influent concentration, and filter media sizes could not be fitted with theoretical curves using the same values of $\alpha_p\beta$ and β'. This shows the necessity of establishing an empirical relationship among $\alpha_p\beta$, β', and different operating parameters from the limited number of laboratory-scale radial flow experiments in order to simulate the concentration and headloss profiles for different operating conditions.

NOTATION

A	= surface area of the filter, L^2
a_g	= specific surface of filter medium, L^{-1}
B	= permeability, LT^{-1}
c_o	= influent suspended solids concentration, ML^{-3}
c^*	= effluent suspended solids concentration, ML^{-3}
d_c	= diameter of the filter grain, L
d_p	= diameter of particles in suspension, L
f	= porosity
g	= acceleration due to gravity, LT^{-2}
H	= thickness of filter medium, L
K	= Kozney's constant
ΔL	= thickness of filter layer considered, L
N	= number of particle collectors, L^{-3}
N_c	= number of filter grains, L^{-3}
N_p	= total number of particles deposited, L^{-3}
n	= number concentration of suspended particles in the effluent, L^{-3}
n_o	= number concentration of suspended particles in the influent, L^{-3}
Q	= influent flow rate, L^3T^{-1}
r	= radius of filter, L
r_o	= radius of influent layer, L
S	= sphericity coefficient
t	= filtration time, T

v = filtration velocity, LT^{-1}
v_o = approach velocity, LT^{-1}

Greek Letters

α = particle-to filter grain attachment coefficient
α_p = particle-to-particle attachment coefficient
β = fraction of retained particles that act as particle collectors
β' = coefficient in headloss equation
η = contact efficiency of a filter grain
η_p = contact efficiency of a particle collector
η_r = removal efficiency of a single collector
μ = viscosity of the suspension, $ML^{-1}T^{-1}$
ρ_l = density of the liquid, ML^{-3}

LITERATURE CITED

Ives, K. J., "Advances in Deep Bed Filtration," *Trans. Inst. Chem. Eng.*, **48**, T94 (1971).

O'Melia, C. R., and W. Ali, "The Role of Retained Particles in Deep Bed Filtration," *Prog. Water Technol.*, **10**, 123 (1978).

Tobiason, J. E., "Packed Bed Filtration: Experimental Investigation and Conceptual Analysis of a Filter Ripening Model," M.S. Thesis, Univ. North Carolina, Chapel Hill (1979).

Vigneswaran, S., "Contribution à la Modèlization de la Filtration dans la Masse," Docteur Ingenieur Thesis, Montpellier, France (1980).

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